Math 128A: Homework 4

Due: July 12

1. Suppose $f \in C^{2n+2}([a,b])$ and $x_0, x_1, \ldots, x_n$ are distinct points in $[a,b]$. Let $H(x)$ be the Hermite interpolating polynomial for f with respect to these points. Prove that for all $x \in [a,b]$, there exists $c(x) \in (a,b)$ such that

$$f(x) - H(x) = \frac{f^{(2n+2)}(c(x))}{(2n+2)!} \Pi_{i=0}^{n} (x - x_i)^2.$$

Hint: use the same method as in the Lagrange polynomial error estimate on the function

$$g(t) = f(t) - H(t) - (f(x) - H(x)) \frac{\Pi_{i=0}^{n} (t - x_i)^2}{\Pi_{i=0}^{n} (x - x_i)^2}$$

along with the fact that g' has distinct $2n+2$ zeros in $[a,b]$.

2. Find the Hermite interpolating polynomial for the following functions with respect to the given points.
   (a) $f(x) = e^{-x^2}$ with respect to $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.
   (b) $f(x) = x|x|$ with respect to $x_0 = -3$, $x_1 = 0$ and $x_2 = 2$.

3. Consider the following dataset.

x_i	0.45	0.5525	0.8225	1.285	1.755
$f(x_i)$	−0.175155	−0.157725	−0.0737543	0.130069	0.326422

   (a) Use Neville's method to find the best degree one, two, three and four approximations to $f(1.1825)$. (**Hint:** you may have to rearrange the order of the points)
   (b) Find the general interpolating polynomial of degree four for f in the Newton form.

4. Consider the differentiation formulae

(i)

$$f'(x_0) = \frac{-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)}{2h} + \frac{h^2}{3} f^{(3)}(c(x_0))$$

(ii)

$$f''(x_0) = \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2} - \frac{h^2}{12} f^{(4)}(\hat{c}(x_0)).$$

For both (i) and (ii)

(a) find the optimal $h > 0$ that minimizes the error bound in the derivative approximation. You may assume that the appropriate higher derivatives of f are bounded.

(b) For the optimal value of h, find the actual error for $f(x) = \sin(x)$ at $x_0 = \pi/6$.